

Exact solution of the asymmetric exclusion model with particles of arbitrary size

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A generalization of the simple exclusion asymmetric model is introduced. In this model an arbitrary mixture of molecules with distinct sizes $s=0,1,2,\dots$, in units of lattice space, diffuses asymmetrically on the lattice. A related surface growth model is also presented. Variations of the distribution of the molecules sizes may change the excluded volume almost continuously. We solve the model exactly through the Bethe ansatz and the dynamical critical exponent z is calculated from the finite-size corrections of the mass gap of the related quantum chain. Our results show that for an arbitrary distribution of molecules, the dynamical critical behavior is on the Kardar-Parisi-Zhang universality. [S1063-651X(99)01607-4]

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I. INTRODUCTION

The asymmetric simple exclusion model [1] is a one-dimensional stochastic model that describes the time fluctuations of particles diffusing asymmetrically on the lattice. If we interpret an occupied site as $\sigma_i^z = +1$ and a vacant site as $\sigma_i^z = -1$, the time evolution of the probability distribution of particles is given by the following asymmetric XXZ Hamiltonian:

$$H = - \sum_{i=1}^N \left[\epsilon_+ \sigma_j^- \sigma_{j+1}^+ + \epsilon_- \sigma_j^+ \sigma_{j+1}^- + \frac{1}{4} (1 - \sigma_j^z \sigma_{j+1}^z) \right], \quad (1)$$

where N is the number of lattice sites and $\sigma_i^\pm = (\sigma^x \pm i\sigma^y)/2$ are the raising and lowering Pauli operators. Periodic boundary conditions are imposed and ϵ_+ and ϵ_- ($\epsilon_+ + \epsilon_- = 1$) are the transition probabilities for having a motion to the right and to the left, respectively. The physical properties of this non-Hermitian quantum chain as well as the related asymmetric six-vertex model are still under extensive investigations [2–4]. This model also describes the surface fluctuations in a growth model known as a single-step model [5,6] where $(\sigma_i^z + \sigma_{i+1}^z)/2 = -1, 0, 1$ is the height difference between nearest-neighbor steps located at the odd-half-integer sites $(i - 1/2)$ ($i = 1, 2, \dots$). The master equation defining the Hamiltonian (1) can also be interpreted [7] as the discretized version of the noisy Burgers equation or the Kardar-Parisi-Zhang (KPZ) equation [8] governing the motion of the height of growing surfaces whose local growth velocity depends nonlinearly on the local shape. The connection between the scaling behavior of the structure function [6,9] of the stochastic model and the finite-size dependence of the real part of the mass gap G_N gives us the dynamical critical exponent z ,

$$\text{Re}(G_N) \sim N^{-z}.$$

This connection enabled Gwa and Spohn [7] to explore the Bethe-ansatz solution of Eq. (1) and calculate exactly the exponent $z = \frac{3}{2}$ in the highly anisotropic limit $\epsilon_- = 0$. Subsequently, Kim [10] extended this result for $\epsilon_- > 0$.

In a previous paper [11] we observed, in connection with a model for a strongly correlated system, that it is possible to keep the exact integrability of the XXZ chain by enlarging the excluded volume to the spins. Motivated by these results we introduce in this paper a generalization of the simple exclusion model where each particle, individually instead of having size 1, in units of lattice spacing, may have an arbitrary and distinct integer size ($s_i = 0, 1, 2, \dots$). Particles with size $s_i > 1$ will produce a stronger excluded volume than those in the simple exclusion model where all the particles have unity size ($s_1 = s_2 = \dots = s_n = 1$). Particles with size zero ($s = 0$) produce no excluded volume since we may put an arbitrary number of them at the same site. By considering arbitrary mixtures of molecules with appropriate sizes, we may change continuously the excluded volume in the bulk limit. The time-evolution operator for these models are generalizations of the ferromagnetic XXZ chain where restrictions on spin configurations, which depend on the particular sizes of the molecules s_i , are added. We show that for arbitrary distribution of the molecule's sizes the eigenspectrum of the related Hamiltonian can be calculated exactly through the Bethe ansatz. The exact integrability for the particular case where all the molecules have the same size and the diffusion is fully asymmetric was also verified recently by Sasamoto and Wadati [12].

Following Gwa and Spohn [7] we show, in the anisotropic limit $\epsilon_- = 0$, that for arbitrary distribution of molecules the real part of the gap behaves as $\text{Re}(G) \sim N^{-3/2}$, giving a universal KPZ behavior with dynamical critical exponent $z = \frac{3}{2}$. Since in our model the excluded volume can be controlled continuously by changing the distribution of molecules, the above exact results imply that the exclusion volume effect is irrelevant to the KPZ dynamics.

The paper is organized as follows. In the next section we introduce the generalized asymmetric model and the related

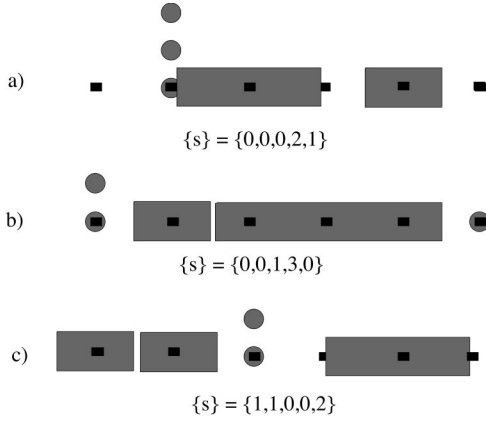


FIG. 1. Example of configurations of molecules with distinct sizes in the asymmetric diffusion problem. These examples correspond to $n=5$ particles in a lattice with size $N=6$. The sizes $\{s_1, s_2, \dots, s_5\}$ are shown in (a)–(c).

generalized surface growth model. The Bethe-ansatz solution of our model is presented in Sec. III and in Sec. IV a numerical and analytical calculation of the dynamical critical exponent z is presented. Finally, in Sec. V we give our conclusions and in the Appendix we relate exactly the general asymmetric exclusion model with several boundary conditions with the simple exclusion model, in different lattice sizes.

II. THE GENERALIZED ASYMMETRIC EXCLUSION MODEL

The simplest realization of the model we consider in this paper is the asymmetric diffusion of molecules (or particles) on a lattice of size N , where each molecule's i ($i=1,2,\dots,n$) may have a distinct size $s_i=0,1,2,\dots$, in units of lattice spacing. We represent the molecules on the lattice by placing their center of mass at the lattice sites. In Fig. 1 we show some examples of configurations of $n=5$ molecules with the size's distribution $\{s\}$ in a lattice with $N=6$ sites. Molecules of size $s=0$ are special since in a given lattice point we can put an arbitrary number of them. As we can see from Fig. 1, the minimum distance between two particles with size s, s' on the lattice is given by

$$d_{s,s'} = \text{Int}\left(\frac{s+s'+1}{2}\right), \quad s, s' = 0, 1, 2, \dots, \quad (2)$$

where $\text{Int}(x)$ is the integer part of x . In order to describe the occupancy of a given site i ($i=1,2,\dots,N$), we attach on it a site variable β_i taking integer values ($\beta_i \in \mathbb{Z}$). If $\beta_i=0$, the site is empty; on the other hand if $\beta_i>0$, we have a molecule of size $s=\beta_i$ and if $\beta_i=-n<0$ we have n molecules of size 0. The allowed configurations $\{\beta_i\}=\{\beta_1, \beta_2, \dots, \beta_N\}$ are those satisfying the constraints imposed by the sizes of molecules in a periodic lattice, i.e., if $\beta_i \neq 0$ and $\beta_j \neq 0$ we must have $|i-j| \geq d_{s(\beta_i), s(\beta_j)}$, where $s(\beta)=0$ if $\beta \leq 0$ and $s(\beta)=\beta$ if $\beta > 0$ is the excluded volume (or size) associated to β .

The master equation for the probability distribution $P(\{\beta\}, t)$ can be written in general as

$$\frac{\partial P(\{\beta\}, t)}{\partial t} = -\Gamma(\{\beta\} \rightarrow \{\beta'\}) P(\{\beta\}, t) + \Gamma(\{\beta'\} \rightarrow \{\beta\}) P(\{\beta'\}, t), \quad (3)$$

where $\Gamma(\{\beta\} \rightarrow \{\beta'\})$ is the transition rate, where a configuration $\{\beta\}$ changes to $\{\beta'\}$. In the model under consideration we only allow, whenever it is possible, a single particle to diffuse into its nearest-neighbor sites. The possible motions are diffusion to the right,

$$\begin{aligned} \beta_i \emptyset_{i+1} &\rightarrow \emptyset_i \beta_{i+1}, & \beta > 0, \\ \beta_i \gamma_{i+1} &\rightarrow (\beta+1)_i (\gamma-1)_{i+1}, & \beta < 0, \gamma \leq 0, \end{aligned} \quad (4)$$

with transition rate ϵ_R , and diffusion to the left,

$$\begin{aligned} \emptyset_i \beta_{i+1} &\rightarrow \beta_i \emptyset_{i+1}, & \beta > 0, \\ \gamma_i \beta_{i+1} &\rightarrow (\gamma-1)_i (\beta+1)_{i+1} \beta < 0, & \gamma \leq 0, \end{aligned} \quad (5)$$

with transition rate ϵ_L . The master equation (2) can be written as a Schrödinger equation in Euclidean time (see Ref. [13] for general application for two-body processes),

$$\frac{\partial |P\rangle}{\partial t} = -H|P\rangle, \quad (6)$$

if we interpret $|P\rangle \equiv P(\{\beta\}, t)$ as the associated wave function. If we represent β_i as $|\beta\rangle_i$, the vector $|\beta\rangle_1 \otimes |\beta\rangle_2 \otimes \dots \otimes |\beta\rangle_N$ will give us the associated Hilbert space. The process (4) and (5) gives us the Hamiltonian

$$\begin{aligned} H &= -D P \sum_{i=1}^N (H_{i,i+1}^> + H_{i,i+1}^<) P, \\ H_{i,j}^> &= \sum_{\beta=1}^{\infty} [\epsilon_+ (E_i^{0,\beta} E_j^{\beta,0} - E_i^{\beta,\beta} E_j^{0,0}) \\ &\quad + \epsilon_- (E_i^{\beta,0} E_j^{0,\beta} - E_i^{0,0} E_j^{\beta,\beta})], \\ H_{i,j}^< &= \sum_{\beta=-\infty}^{-1} \sum_{\gamma=-\infty}^0 [\epsilon_+ (E_i^{\beta+1,\beta} E_j^{\gamma-1,\gamma} - E_i^{\beta,\beta} E_j^{\gamma,\gamma}) \\ &\quad + \epsilon_- (E_i^{\gamma-1,\gamma} E_j^{\beta+1,\beta} - E_i^{\gamma,\gamma} E_j^{\beta,\beta})], \end{aligned} \quad (7)$$

with

$$D = \epsilon_R + \epsilon_L, \quad \epsilon_+ = \frac{\epsilon_R}{\epsilon_R + \epsilon_L}, \quad \epsilon_- = \frac{\epsilon_L}{\epsilon_R + \epsilon_L}, \quad (8)$$

and periodic boundary conditions. The matrices $E^{\alpha,\beta}$ are infinite-dimensional with a single nonzero element $(E^{\alpha,\beta})_{i,j} = \delta_{\alpha,i} \delta_{\beta,j} (\alpha, \beta, i, j \in \mathbb{Z})$. The projector P keeps on the Hilbert space only the vectors $|\{\beta\}\rangle$ satisfying the constraint (2), which mathematically means that for all $\beta_i, \beta_j \neq 0$, $|i-j| \geq d_{s(\beta_i), s(\beta_j)}$. The constant D in Eq. (7) fixes the time scale and for simplicity we chose $D=1$. A particular simplification of the above Hamiltonian occurs when all the molecules have the same fixed size $s>0$. In this case the Hamiltonian can be expressed in terms of spin- $\frac{1}{2}$ Pauli matrices,

$$H_{\{s_1=\dots=s_n=s\}} = -P_s \left(\sum_{i=1}^N [\epsilon_+ \sigma_i^- \sigma_{i+1}^+ + \epsilon_- \sigma_i^+ \sigma_{i+1}^- + \frac{1}{2}(\epsilon_+ + \epsilon_-)(\sigma_i^z \sigma_{i+1}^z - 1)] \right) P_s, \quad (9)$$

where now P_s projects out configurations where two up spins, in a σ^z basis, are at a distance smaller than the size $s > 0$. In the case where $s=1$, the projector $P_s=1$ and we have the standard asymmetric exclusion Hamiltonian (1). In terms of Pauli matrices this operator has the general form

$$P_s = \prod_i \left[\frac{1}{2}(1 - \sigma_i^z) + \frac{1}{2}(1 + \sigma_i^z) \prod_{l=1}^{s-1} \left(\frac{1 - \sigma_{i+l}^z}{2} \right) \right]. \quad (10)$$

The Hamiltonian (9) can be more easily compared with standard magnetic quantum chains by performing for $\epsilon_+, \epsilon_- \neq 0$ the following canonical change of variables:

$$\sigma_i^\pm \rightarrow \left(\frac{\epsilon_-}{\epsilon_+} \right)^{\pm i/2} \sigma_i^\pm, \quad \sigma^z \rightarrow \sigma^z, \quad i=1,2,\dots,N, \quad (11)$$

which gives

$$H = -\frac{1}{2} \sqrt{\epsilon_+ \epsilon_-} \sum_{i=1}^N P_s [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta (\sigma_i^z \sigma_{i+1}^z - 1)] P_s, \quad (12)$$

$$\Delta = \frac{\epsilon_+ + \epsilon_-}{\sqrt{\epsilon_R \epsilon_L}}.$$

Apart from the projector P_s , this is the ferromagnetic XXZ chain or the anisotropic Heisenberg chain. However, in distinction with Eq. (8) the boundary conditions are now twisted,

$$\sigma_{N+1}^\pm = \left(\frac{\epsilon_+}{\epsilon_-} \right)^{\pm N/2} \sigma_1^\pm, \quad \sigma_{N+1}^z = \sigma_1^z. \quad (13)$$

We expect that ferromagnetic quantum chains like those in Eq. (12) are gapped for $\Delta > 1$. However, since $(\epsilon_+/\epsilon_-)^{N/2} \rightarrow \infty$, for $N \rightarrow \infty$ the boundary condition gives us interaction with the same degree of importance as the totality of the other interactions (see [14] for a related problem). As we will see, from the exact solution of Eqs. (7) and (9), this surface term is strong enough to produce a gapless eigenspectrum.

In surface growth physics, the asymmetric simple exclusion model is related to the single-step model. Similarly, our generalized model is also related to a generalization of the single-step model. The surface configurations in this growth model are obtained by defining height variables h_i ($i=1,2,\dots$), which are related to the spin variables in our generalized asymmetric diffusion model. For simplicity, we are going to present only the surface growth model related to the diffusion problem where all molecules have the same size s . Let us consider initially $s > 0$. For a given configuration $\{\beta_1, \dots, \beta_N\}$ of molecules of size s , the height variables should obey

$$h_{i+1} - h_i = f(\beta_{i-s/2}, \beta_{i-s/2+1}, \dots, \beta_{i+s/2}, \beta_{i+s/2+1}), \quad (14)$$

where $f=0$ for all allowed configurations except in the case

$$f(0, \dots, 0) = -1 \quad \text{and} \quad f(s, 0, \dots, 0, s) = 1. \quad (15)$$

The variables $\{\beta\}$ of the related diffusion model are defined at the links or at the same positions of the height variables $\{h\}$ depending on if the size of the molecules s is odd or even, respectively. The number of molecules n ($0,1,\dots$) in the generalized asymmetric diffusion is conserved and for each value of n we are going to have a growth model with different boundary conditions in the spacial direction. The dynamical rules defining the growth are the following.

(a) No steps on the surface are allowed to be higher than 1, in units of lattice spacing in the growth direction, i.e.,

$$h_{i+1} - h_i = 1, 0, -1 \quad (i=1, \dots, n-1). \quad (16)$$

(b) All the local valleys and hills should have a size, in units of the lattice spacing in the spatial direction, which is a multiple of $b = s + 1$.

(c) The following boundary condition should be satisfied:

$$h_{i+N} = h_i - \bar{h}, \quad \bar{h} = b\{[N/b]_I - n\} + [N/b]_R, \quad (17)$$

where by $[N/b]_I$ and $[N/b]_R$ we mean the integer part and the rest of the division N/b .

(d) The surface changes whenever, by obeying the previous requirements, we can still adsorb ($h_{i+l} \rightarrow h_{i+l} + 1, l=0,1,\dots,b-1$) or desorb ($h_{i+l} \rightarrow h_{i+l} - 1, l=0,1,\dots,b-1$) at height h_i ($i=1,\dots,N$) a rectangular brick of size b in the spatial direction and size 1 in the growth direction.

We choose a height h_i ($i=1,\dots,n$) at random. If it is possible to adsorb or desorb a brick, with probability $\epsilon_+/2$ ($\epsilon_-/2$) we desorb (adsorb) a brick, and do nothing with probability $\frac{1}{2}$. If it is possible, at h_i , only to desorb (adsorb) a brick, with probability ϵ_+ (ϵ_-) we desorb (adsorb) a brick and with probability $1 - \epsilon_+$ ($1 - \epsilon_-$) we do nothing. In Fig. 2, we show for $N=7$, $s=2$ ($b=3$), and $n=2$ some examples of the possible configurations of the surface. In this figure we also show the corresponding particle configurations in the diffusion problem. We can verify that for arbitrary s (or b), as long as the growth model is not periodic ($N \neq nb$) there exists an exact one-to-one correspondence between the configurations of particles and those of the surface, with the transitions between them described by the Hamiltonian (7). On the other hand, if the growth model is periodic ($N = nb$), there exist b configurations in the asymmetric diffusion problem that correspond to a single surface configuration (the flat surface). Consequently, the Hamiltonian (7) does not describe exactly the generalized step model in a finite lattice. However, as N increases, this difference decreases and Eq. (7) also describes the fluctuations of the growth model.

Finally, in the case where all the molecules have size zero, a possible growth model is obtained by defining the height variable h_i ($i=1,2,\dots,N$) at the same position of the molecules in the diffusion problem. For a given configuration $\{n_1, n_2, \dots, n_N\}$ with n_i molecules at sites i , the height variables in the surface model ($h_{i+1} \geq h_i$) satisfy

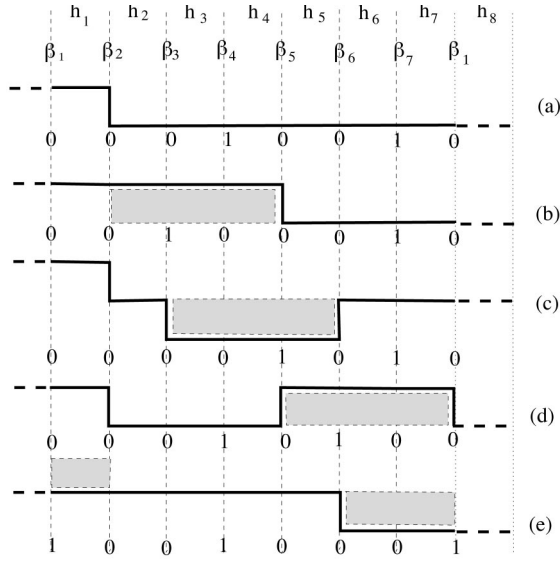


FIG. 2. Examples of configurations in the growth model with $N=7$, $b=3$, and $n=2$. The corresponding configurations of particles in the asymmetric diffusion problem [Eq. (7)] are also shown. The configurations (b) and (d) are obtained by adsorbing a brick (size $b=3$) or by moving a particles to the left in configuration (a), and configurations (c) and (e) are obtained by desorbing a brick, or by moving a particle to the right in configuration (a).

$$h_i - h_{i-1} = n_i, \quad i = 2, \dots, N, \quad (18)$$

with the boundary condition

$$h_{N+1} = h_1 + n, \quad (19)$$

where $n = \sum n_i$ is the total number of molecules. Bricks of unity size are desorbed (adsorbed) with transition rates ϵ_+ (ϵ_-) if the final configuration satisfies $h_{i+1} \geq h_i$ ($i = 1, 2, \dots, N-1$).

III. THE BETHE-ANSATZ EQUATIONS

We present in this section the exact solution of the general quantum chain (7). For simplicity, let us consider initially the case where all the molecules have the same size s ($0, 1, \dots$). In the particular case where $s=1$, we have the standard simple exclusion model whose Bethe-ansatz solution was obtained by Gwa and Spohn [7] and can also be obtained after the canonical transformation (13) from the Bethe-ansatz solution of the XXZ chain with twisted boundary conditions [15]. The exact integrability of the fully asymmetric version of Eq. (12) ($\epsilon_- = 0$), for $s > 0$, was verified directly in the master equation by Sasamoto and Wadati [12], and the model with $s=0$ is related to the limit $q \rightarrow \infty$ of the q -boson hopping model introduced by Bogoliubov *et al.* [16,17].

Due to the conservation of particles in the diffusion processes, the total number of particles are good quantum numbers and we can separate the associated Hilbert space into block-disjoint sectors labeled by the number n of particles. We therefore consider the eigenvalue equation

$$H|n\rangle = E|n\rangle, \quad (20)$$

where

$$|n\rangle = \sum_{\{x\}} f(x_1, \dots, x_n) |x_1, \dots, x_n\rangle. \quad (21)$$

Here x_1, \dots, x_n denotes the location of particles on the chain and the summation extends over all sets $\{x\}$ of the n nondecreasing integers satisfying

$$x_{i+1} \geq x_i + s, \quad i = 1, \dots, n-1, \quad s \leq x_n - x_1 \leq N - s. \quad (22)$$

It is important to notice that some of these coordinates may coincide in the case where the particles have zero size.

$n=1$. For one particle on the chain as a consequence of the translational invariance of Eq. (7), the eigenfunctions are the momentum- k eigenfunctions,

$$|1\rangle = \sum_{x=1}^N f(x) |x\rangle, \quad f(x) = e^{ikx}, \quad (23)$$

where

$$k = \frac{2\pi}{N}l; \quad l = 0, 1, \dots, N-1, \quad (24)$$

and energy given by

$$E = e(k) \equiv -(\epsilon_- e^{ik} + \epsilon_+ e^{-ik} - 1). \quad (25)$$

$n=2$. For two particles on the lattice the eigenvalue equation (20) gives us two distinct relations depending on the relative position of the particles. If the two particles are at positions x_1 and x_2 satisfying $x_2 > x_1 + s$, we obtain

$$E f(x_1, x_2) = -\epsilon_+ f(x_1 - 1, x_2) - \epsilon_- f(x_1 + 1, x_2) + 2f(x_1, x_2) - \epsilon_+ f(x_1, x_2 - 1) - \epsilon_- f(x_1, x_2 + 1), \quad (26)$$

which can be solved by the ansatz

$$f(x_1, x_2) = e^{ik_1 x_1} e^{ik_2 x_2}, \quad (27)$$

which gives

$$E = e(k_1) + e(k_2). \quad (28)$$

Since the relation (28) is symmetric in k_1 and k_2 , we can write a more general solution of Eq. (26) as

$$f(x_1, x_2) = A_{12} e^{ik_1 x_1} e^{ik_2 x_2} + A_{21} e^{ik_2 x_1} e^{ik_1 x_2} \quad (29)$$

with the same energy as in Eq. (28). The second relation happens when $x_2 = x_1 + s$. In this case we have

$$E f(x_1, x_1 + s) = -\epsilon_+ f(x_1 - 1, x_1 + s) - \epsilon_- f(x_1, x_1 + s + 1) + f(x_1, x_1 + s). \quad (30)$$

If we now substitute the ansatz (29) with the energy (28), the constants A_{12} and A_{21} , initially arbitrary, should now satisfy

$$\frac{A_{12}}{A_{21}} = - \left(\frac{e^{ik_1}}{e^{ik_2}} \right)^{s-1} e^{i\Psi_{12}}, \quad (31)$$

$$e^{i\Psi_{jl}} = \frac{\epsilon_+ + \epsilon_- e^{i(k_j+k_l)} - e^{ik_j}}{\epsilon_+ + \epsilon_- e^{i(k_j+k_l)} - e^{ik_l}}. \quad (32)$$

The ‘‘wave numbers’’ k_1 and k_2 are complex in general and are fixed due to the cyclic boundary condition

$$f(x_2, x_1+N) = f(x_1, x_2), \quad (33)$$

which from Eq. (29) give us the equations

$$\frac{A_{12}}{A_{21}} = e^{-ik_2N}, \quad \frac{A_{21}}{A_{12}} = e^{-ik_1N}. \quad (34)$$

Equations (31) and (32) give us the Bethe-ansatz equations for $n=2$,

$$e^{ik_jN} = - \prod_{l=1}^2 \left(\frac{e^{ik_j}}{e^{ik_l}} \right)^{s-1} e^{i\Psi_{j,l}}, \quad j=1,2, \quad (35)$$

with energy given by Eq. (28).

General n. The above calculation can be generalized for arbitrary values of n . The ansatz for the wave function becomes

$$f(x_1, \dots, x_n) = \sum_P A_{P_1, P_2, \dots, P_n} e^{i(k_{P_1}x_1 + \dots + k_{P_n}x_n)}, \quad (36)$$

where the sum extends over all permutations P of $1, 2, \dots, n$. If $x_{i+1} - x_i > s$ for $i=1, 2, \dots, n$, it is easy to see that the eigenvalue equation (20) is satisfied by the ansatz (36) with energy

$$E = \sum_{j=1}^n e(k_j). \quad (37)$$

If a pair of particles is at positions x_i, x_{i+1} , where $x_{i+1} = x_i + s$, Eq. (20) with the ansatz (36) and the relation (37) give us

$$\frac{A_{P_1, \dots, P_i, P_{i+1}, \dots, P_n}}{A_{P_1, \dots, P_{i+1}, P_i, \dots, P_n}} = - e^{i(s-1)(k_{P_i} - k_{P_{i+1}})} e^{i\Psi_{P_i, P_{i+1}}}. \quad (38)$$

Inserting the ansatz (36) in the boundary condition

$$f(x_2, \dots, x_n, x_1+N) = f(x_1, \dots, x_n), \quad (39)$$

we obtain the additional relation

$$A_{P_1, \dots, P_n} = e^{ik_{P_1}N} A_{P_2, \dots, P_n, P_1}. \quad (40)$$

If we iterate the relation (38) n times, Eq. (40) gives us the Bethe-ansatz equations

$$e^{ik_jN} = (-)^{n-1} \prod_{l=1}^n \left(\frac{e^{ik_j}}{e^{ik_l}} \right)^{s-1} \frac{\epsilon_+ + \epsilon_- e^{i(k_j+k_l)} - e^{ik_j}}{\epsilon_+ + \epsilon_- e^{i(k_j+k_l)} - e^{ik_l}} \quad (41)$$

for $j=1, 2, \dots, n$. The solutions $\{k_j\}$ of these equations with Eq. (37) give us the eigenenergies of the Hamiltonian (7). Furthermore, it follows from a lattice shifting that the wave

functions given by the ansatz (36) are also eigenfunctions of the momentum operator \hat{P} with eigenvalue

$$p = \sum_{j=1}^n k_j \bmod(2\pi) = \frac{2\pi}{N} l, \quad l=0, 1, \dots, N-1. \quad (42)$$

In the particular case where $s=1$, Eqs. (37), (41), and (42) recover the results presented in Ref. [7].

Let us now consider the general case where we have n molecules with arbitrary given sizes $\{s_1, s_2, \dots, s_n\}$ ($s_i = 0, 1, 2, \dots$) and the Hamiltonian given by Eq. (7). In this case each type of molecule is conserved separately. Moreover, since in the diffusion processes the particles only interchange positions with the vacant sites, a given order $\{s_1, s_2, \dots, s_n\}$ of particles remains conserved up to cyclic permutations. The wave functions can be written as

$$|s_1, s_2, \dots, s_n\rangle = \sum_{\{c\}} \sum_{\{x\}} f^{s_{c_1}, \dots, s_{c_n}}(x_1, \dots, x_n) |x_1, \dots, x_n\rangle. \quad (43)$$

Here $f^{s_{c_1}, \dots, s_{c_n}}(x_1, \dots, x_n)$ is the amplitude of a configuration where particles of sizes s_1, \dots, s_n occupy the positions x_1, \dots, x_n , respectively. The summation $\{c\}$ extends over all cyclic permutations $\{c_1, \dots, c_n\}$ of integers $\{1, \dots, n\}$, and the summation $\{x\}$ extends, for a given distribution $\{s_{c_1}, \dots, s_{c_n}\}$ of molecules, to increasing integers satisfying

$$x_{i+1} \geq x_i + d_{s_{c_i}, s_{c_{i+1}}}, \quad i=1, \dots, n-1, \\ d_{s_{c_n}, s_{c_1}} \leq x_n - x_1 \leq N - d_{s_{c_n}, s_{c_1}}. \quad (44)$$

The ansatz that we expect to be valid, which replaces Eq. (29), is

$$f^{s_1, \dots, s_n}(x_1, \dots, x_n) = \sum_P A_{P_1, \dots, P_n}^{s_1, \dots, s_n} e^{i(k_{P_1} + \dots + k_{P_n})}, \quad (45)$$

where $A_{k_{P_1}, \dots, k_{P_n}}^{s_1, \dots, s_n}$ and $\{k_1, \dots, k_n\}$ are going to be fixed by imposing that Eq. (44) with Eq. (45) is a solution of the eigenvalue equation of the general Hamiltonian (7).

Let us consider the eigenvectors of Eq. (7) with a different number of particles.

$n=1$. For one particle on the chain the ansatz (45) coincides with Eq. (36) and the wave functions and energies are given by Eqs. (23) and (25), respectively.

$n=2$. If both particles are identical, $s_1 = s_2 = s$, we have the same situation considered previously in Eqs. (26)–(35). The wave functions $|s, s\rangle$ are given by Eq. (45) with energies given by Eqs. (28) and (35). However, if the particles are distinct, the situation is different. If the particles are located at positions x_1 and x_2 , with $x_2 - x_1 > d_{s_1, s_2}$, the ansatz (45) is valid with energy given by Eq. (28) and no restrictions on $\{A_{k_{P_1}, k_{P_2}}^{\alpha_i, \alpha_j}\}$ are necessary. If the particles are at the closest distance $x_2 = x_1 + d_{s_1, s_2}$, Eq. (30) is replaced by

$$\begin{aligned}
Ef^{s_1, s_2}(x_1, x_1 + d_{s_1, s_2}) &= -\epsilon_+ f^{s_1, s_2}(x_1 - 1, x_1 + d_{s_1, s_2}) \\
&\quad - \epsilon_- f^{s_1, s_2}(x_1, x_1 + d_{s_1, s_2} + 1) \\
&\quad + f^{s_1, s_2}(x_1, x_1 + d_{s_1, s_2}). \quad (46)
\end{aligned}$$

Inserting in the above equation the ansatz (45) and the energy (28), we obtain the relation

$$A_{P_1, P_2}^{s_1, s_2} = -e^{i\Psi_{P_1, P_2}} \sum_{s'_1, s'_2} S_{s'_2, s'_1}^{s_1, s_2}(k_{P_1} - k_{P_2}) A_{P_2, P_1}^{s'_1, s'_2}, \quad (47)$$

where Ψ_{P_1, P_2} is given by Eq. (32) and the elements of the S matrix are given by

$$S_{\gamma, \mu}^{\alpha, \beta}(k) = e^{i(d_{\alpha, \beta} - 1)k} \delta_{\alpha, \mu} \delta_{\beta, \gamma}. \quad (48)$$

The wave numbers k_1 and k_2 are going to be fixed by the boundary condition

$$f^{s_2, s_1}(x_2, x_1 + N) = f^{s_1, s_2}(x_1, x_2), \quad (49)$$

but instead of deriving the Bethe-ansatz equations for $n=2$ let us consider the case of general n .

General n . The ansatz (45) applied to the case where two particles are at their closest distance gives us the generalization of Eq. (47),

$$\begin{aligned}
A_{\dots, P_1, P_2, \dots}^{\alpha, \beta, \dots} &= -e^{i\Psi_{P_1, P_2}} \sum_{\alpha', \beta'} S_{\alpha', \beta'}^{\alpha, \beta} \\
&\quad \times (k_{P_1} - k_{P_2}) A_{\dots, P_2, P_1, \dots}^{\beta', \alpha', \dots}, \quad (50)
\end{aligned}$$

with S given by Eq. (48). Successive applications of this equation give us in general different relations between the amplitudes. For example, $A_{\dots, k_3, k_2, k_1, \dots}^{\alpha, \beta, \gamma, \dots}$ relate to $A_{\dots, k_3, k_2, k_1, \dots}^{\gamma, \beta, \alpha, \dots}$ by performing the permutations $\alpha\beta\gamma \rightarrow \beta\alpha\gamma \rightarrow \gamma\beta\alpha$ or $\alpha\beta\gamma \rightarrow \alpha\gamma\beta \rightarrow \gamma\alpha\beta \rightarrow \gamma\beta\alpha$, and consequently the S matrix should satisfy the Yang-Baxter [18,19] equation

$$\begin{aligned}
&\sum_{\gamma, \gamma', \gamma''} S_{\gamma, \gamma'}^{\alpha, \alpha'}(k_1 - k_2) S_{\beta, \gamma''}^{\gamma, \alpha''}(k_1 - k_3) S_{\beta', \beta''}^{\gamma', \gamma''}(k_2 - k_3) \\
&= \sum_{\gamma, \gamma', \gamma''} S_{\gamma', \gamma''}^{\alpha', \alpha''}(k_2 - k_3) S_{\gamma, \beta''}^{\alpha, \gamma''}(k_1 - k_3) S_{\beta, \beta'}^{\gamma, \gamma'}(k_1 - k_2). \quad (51)
\end{aligned}$$

Actually the relation (51) is a necessary and sufficient condition [18,19] to obtain a nontrivial solution for the amplitudes in Eq. (50). The validity of Eq. (51) can easily be verified for the diagonal S matrix (48) in the problem we are considering.

The boundary condition

$$f^{s_1, \dots, s_n}(x_1, \dots, x_n) = f^{s_2, \dots, s_n, s_1}(x_2, \dots, x_n, x_1) \quad (52)$$

implies the relation between the amplitudes

$$A_{P_1, \dots, P_n}^{s_1, \dots, s_n} = e^{ik_{P_1}N} A_{P_2, \dots, P_n, P_1}^{s_2, \dots, s_n, s_1}. \quad (53)$$

If we now apply relation (50) n times, we can obtain a relation between the amplitudes with the same momenta, i.e.,

$$\begin{aligned}
A_{P_1, \dots, P_n}^{s_1, \dots, s_n} &= (-)^{n-1} e^{i \sum_{l=2}^n \Psi_{P_l, P_1}} e^{ik_{P_1}N} \\
&\quad \times \sum_{\{s'_1, \dots, s'_n\}} \sum_{\{s''_1, \dots, s''_n\}} S_{s'_1, s'_1}^{s_1, s_2''}(k_{P_1} \\
&\quad - k_{P_1}) S_{s'_2, s'_2}^{s_2, s_3''}(k_{P_2} - k_{P_1}) \cdots S_{s'_n, s'_n}^{s_n, s_1''}(k_{P_n} \\
&\quad - k_{P_1}) A_{P_1, \dots, P_n}^{s'_1, \dots, s'_n}, \quad (54)
\end{aligned}$$

where we introduced the extra sum

$$1 = \sum_{s''_2, s''_1} \delta_{s''_2, s'_1} \delta_{s_1, s''_1} = \sum_{s''_2, s''_1} S_{s'_1, s'_1}^{s_1, s_2''}(k_{P_1} - k_{P_1}). \quad (55)$$

In order to fix the values of $\{k_j\}$, we should then find the eigenvalues $\Lambda(k)$ of the matrix

$$\langle \{s\} | T(k) | \{s'\} \rangle = \sum_{\{s''_1, \dots, s''_n\}} \prod_{l=1}^n S_{s'_l, s''_l}^{s_l, s''_{l+1}}(k_{P_l} - k), \quad (56)$$

with $s''_{n+1} = s''_1$. We identify $T(k)$ as the transfer matrix of an inhomogeneous vertex model, with inhomogeneities $(k_{P_l} - k)$, in a periodic lattice. The Boltzmann weights of the vertex models are given by $S_{\gamma\delta}^{\alpha\beta}$ and the number of distinct vertices depends on the particular configuration (type of order) of molecules in our diffusive system. Using Eq. (48), we can see that there exists only one nonzero element for each line, i.e., $\langle s_1, \dots, s_n | T | s_2, \dots, s_n, s_1 \rangle$.

In order to calculate the eigenvalues $\Lambda(k)$, we apply the transfer matrix r times in the state A^{s_1, \dots, s_n} , where r is the minimum number of cyclic rotations of $\{s_1, \dots, s_n\}$, where the configuration repeats the initial one. We may show that

$$T^r A_{P_1, \dots, P_n}^{\alpha_1, \dots, \alpha_n} = \Lambda^r(k) A_{P_1, \dots, P_n}^{\alpha_1, \dots, \alpha_n}. \quad (57)$$

Also it is easy to compute

$$\Lambda^r(k) = \exp \left[i \frac{r}{n} \left(\sum_{l=1}^n d_{s_l, s_{l+1}} - 1 \right) \sum_{l=1}^n (k_l - k) \right]. \quad (58)$$

Finally, substituting $\Lambda(k_{P_l})$ in Eq. (54) we obtain the Bethe-ansatz equations

$$\begin{aligned}
e^{ik_j N} &= e^{i \frac{2\pi}{r} m} (-1)^{n-1} \\
&\quad \times \prod_{l=1}^n e^{i(k_j - k_l)(\tilde{s}-1)} \frac{\epsilon_+ + \epsilon_- e^{i(k_j + k_l)} - e^{ik_j}}{\epsilon_+ + \epsilon_- e^{i(k_j + k_l)} - e^{ik_l}}, \quad (59)
\end{aligned}$$

where $j=1, \dots, n$; $m=0, 1, \dots, r-1$ and

$$\tilde{s} = \frac{1}{n} \sum_{l=1}^n d_{s_l, s_{l+1}} \quad (60)$$

plays the role of an average molecule size of the particular configuration $\{s\}$ of molecules. As we can see, by comparing Eq. (59) with Eq. (41), the extra phase $\exp(i2\pi m/r)(m=0,1,\dots,r-1)$ gives r times more solutions of Eq. (59) than in Eq. (41). This indeed should be the case since the Hilbert space associated to the Hamiltonian (7) of a given distribution of particles of sizes $\{s_1, \dots, s_n\}$, due to the distinguishability of the particles, is r times bigger than that of the Hamiltonian (7) when $s_1=s_2=\dots=s_n$. It is interesting to observe that \tilde{s} can take any non-negative rational number by choosing appropriately $\{s_1, \dots, s_n\}$. Also many distinct distributions of the molecules' sizes with the same effective \tilde{s} will have the same eigenenergies.

In the Appendix we explore our Bethe-ansatz solution to obtain the relationship between the eigenvalues and eigenvectors of the Hamiltonian (7) with different distributions of the molecules' sizes.

IV. THE CRITICAL EXPONENT Z

In this section we calculate the dynamical critical exponent z for the stochastic models presented in Sec. II. This calculation is done by exploiting its connection with the mass gap G_N of the corresponding Hamiltonian,

$$\text{Re}(G_N) \sim N^{-z}. \quad (61)$$

A calculation for arbitrary values of ϵ_+, ϵ_- and density n/N can be done systematically by using the method presented in [10]. However, since universality arguments indicate that z should be independent of the particular values of ϵ_+, ϵ_- , and n , as long as $\epsilon_+ \neq \epsilon_-$, we are going to restrict ourselves, as in [7], to the simplest case where $\epsilon_- = 0, \epsilon_+ = 1$. A general discussion for the other cases, which does not change our results, is given at the end of the Appendix. Defining the variables

$$z_j = 2e^{-ik_j} - 1, \quad (62)$$

the energies (37) and momenta (42) are given by

$$E = \sum_{j=1}^n (1 - z_j)/2, \quad (63)$$

$$e^{-iP} = \prod_{j=1}^n (1 + z_j)/2, \quad (64)$$

respectively. The $\{z_j\}$ variables should satisfy the Bethe-ansatz equations (59),

$$(1 + z_j)^{N - n\tilde{s}} (1 - z_j)^n = -2^N e^{-i(2\pi/r)m} \prod_{l=1}^n \frac{z_l - 1}{(z_l + 1)^{\tilde{s}}}, \quad (65)$$

where $j=1, \dots, n$, $m=0, 1, \dots, r-1$. It is interesting to note that these equations are simpler than the usual Bethe-ansatz equations appearing in other exact integrable systems since the right-hand side of Eq. (65) is independent of the

particular value of j . These equations are even simpler for the special ‘half-filled’ density $\rho = n/N = 1/(1 + \tilde{s})$, i.e.,

$$(1 - z^2)^n = Y, \quad (66)$$

$$Y = -2^{n(1+\tilde{s})} \prod_{l=1}^n \frac{z_l - 1}{(z_l + 1)^{\tilde{s}}}.$$

If we parametrize $Y = -a^n e^{i\theta}$, with $a \geq 0$ and $\theta \in (-\pi/n, \pi/n)$, the $2n$ roots z_j are given by

$$\begin{aligned} z_j &= (1 - y_j)^{1/2}, \quad z_{j+n} = -z_j; \\ y_j &= a e^{i\theta} e^{i2\pi(j-1/2)/n}; \quad j=1, \dots, n. \end{aligned} \quad (67)$$

For a given choice $\{z_j\}$ of the above set and a given value of m ($0, 1, \dots, r-1$), we have only two unknowns, a and θ , which are obtained from the equation

$$(a e^{i\theta})^n = e^{i(2\pi/r)m} 2^{(\tilde{s}+1)n} \prod_{l=1}^n \frac{z_{l(j)} - 1}{[z_{l(j)} + 1]^{\tilde{s}}}. \quad (68)$$

We have solved numerically the above equations for several values of \tilde{s} , m , r , and n . The ground-state energy $E=0$ is obtained by choosing $m=0$ in Eq. (68), and is given by the configuration

$$C_0 = \{z_1, z_2, \dots, z_n\}, \quad (69)$$

with $a = \theta = 0$. In order to find the first excited state, we should consider different choices of $\{z_j\}$ and different values of m . Since $z_j + z_{n+j} = 0$, the energy increases as we take, in the configurations $\{z_j\}$, values of z_j where $n < j \leq 2n$. Therefore, configurations $\{z\}$ associated with low energies are

$$C_1 = \{z_1, z_2, \dots, z_{n-1}, z_{n+1}\} \quad (70)$$

and

$$C_{-1} = \{z_2, \dots, z_{n-1}, z_n, z_{2n}\}. \quad (71)$$

These configurations, from Eq. (64), correspond to states with momentum $-2\pi/N$ and $2\pi/N$, respectively. Our numerical results show that the energy corresponding to the configuration C_0 with $m \neq 0$ behaves for large N as

$$E_{C_0, m} \sim \frac{a}{n^{1/2}} - i \frac{\pi}{r(\tilde{s}+1)} m, \quad m=1, 2, \dots \quad (72)$$

On the other hand, the configuration C_1 or C_{-1} , for sufficiently large values of N , produces the lowest energy when $m=0$, independently of \tilde{s}, r , and behaves as

$$E_{C_{\pm 1}, 0} \sim \frac{a_0}{n^z} + i \frac{b_0}{n^\gamma}, \quad z = \frac{3}{2}, \quad \gamma = 1, \quad (73)$$

where a_0 and b_0 are constants. The energies for different values of m but with configurations $C_{\pm 1}$ also behave similarly to Eq. (73). Comparing Eq. (73) with Eq. (72), we see that the gap is given by $E_{C_{\pm 1}, 0}$ and is real only for the special

TABLE I. Examples of finite-size estimates for the amplitudes a_0 , b_0 and the exponents z and γ in Eq. (73). These estimates correspond to the cases $\tilde{s}=0.25$ and $\tilde{s}=2.5$. The calculated analytical results in the $n \rightarrow \infty$ are also shown in the last line of this table.

| n | $\tilde{s}=0.25$ | | | | $\tilde{s}=2.5$ | | | |
|-------|------------------|-----------|----------|----------|-----------------|----------|----------|----------|
| | a_0 | b_0 | z | γ | a_0 | b_0 | z | γ |
| 10 | 2.522601 | -0.916488 | 1.586388 | 0.934332 | 2.495387 | 0.654527 | 1.576255 | 0.934156 |
| 50 | 2.351517 | -0.941304 | 1.521414 | 0.997428 | 2.345566 | 0.672351 | 1.518526 | 0.997403 |
| 100 | 2.326504 | -0.942187 | 1.510838 | 0.999374 | 2.323540 | 0.672989 | 1.509564 | 0.999367 |
| 150 | 2.318133 | -0.942349 | 1.507253 | 0.999724 | 2.316160 | 0.673106 | 1.506402 | 0.999721 |
| 200 | 2.313942 | -0.942406 | 1.505450 | 0.999846 | 2.312463 | 0.673146 | 1.504811 | 0.999844 |
| 250 | 2.311425 | -0.942432 | 1.504365 | 0.999902 | 2.310243 | 0.673165 | 1.503853 | 0.999900 |
| 300 | 2.309747 | -0.942446 | 1.503640 | 0.999932 | 2.308737 | 0.673175 | 1.503189 | 0.999957 |
| 400 | 2.307648 | -0.942460 | 1.502733 | 0.999962 | | | | |
| 500 | 2.306388 | -0.942466 | 1.502186 | 0.999976 | | | | |
| 800 | 2.304498 | -0.942473 | 1.501364 | 0.999988 | | | | |
| Exact | 2.301346 | -0.942478 | 1.5 | 1 | 2.301346 | 0.673198 | 1.5 | 1 |

case $\tilde{s}=1$, treated in [7]. The values of a and θ that correspond to the first excited state behave asymptotically as

$$a = 1 + \frac{\beta}{n} + o(n^{-1}), \quad \theta = \pm \alpha \frac{(\tilde{s}-1)}{n^{3/2}} + o(n^{-3/2}), \quad (74)$$

where β and α are constants. In order to illustrate our numerical results, we show in Table I the finite-size estimates for the amplitudes a_0, b_0 and the exponents z and γ defined in Eq. (73).

Accepting the behavior (74) for the values of a and θ for the first excited state, we also used the same procedure as in Gwa and Spohn [7] in order to show analytically that $z = \frac{3}{2}$, $a_0 = 2.301\ 345\ 96 \dots$, independently of the value of \tilde{s} and $b_0 = \pi(\tilde{s}-1)/[2(\tilde{s}+1)]$. In the last line of Table I we show the exact results obtained analytically.

These results indicate that all these models with an arbitrary mixture of molecules of different sizes, as well as the corresponding generalized growth models, exhibit a universal behavior with a KPZ-type of dynamical behavior. In the Appendix we show that for general values of ϵ_+ , ϵ_- , and n , the wave functions of Eq. (7) for arbitrary distributions of the molecules' sizes are exactly related. This implies that conditional probabilities and correlation functions for arbitrary distribution $\{s\}$ are exactly related to the simple exclusion problem $\{s_1 = s_2 = \dots = s_n = 1\}$. The eigenvalues of these models are exactly related in the case of free boundaries. In the case in which we have a periodic lattice, the eigenvalues of $H^{\{s\}}$ are exactly related to the asymmetric XXZ chain with twisted boundary condition ϕ proportional to the momentum of the first excited state. Since the momentum of this state is $P = 2\pi/N$, the effect of the twisted angle should not affect the leading order in the mass gap calculations. This implies that for arbitrary values of ϵ_+ , ϵ_- , and density n , the leading-order results of the real part of the gap are the same as those calculated systematically in [10].

V. CONCLUDING REMARKS

We have solved exactly a general asymmetric diffusion problem where the particles may have distinct and arbitrary integer sizes. We also show in Sec. II that these diffusion models are related to generalized growth models. Since through diffusion the particles do not interchange positions, a given order $\{s_1, \dots, s_n\}$ of the distribution of the molecules' sizes on the lattice is fixed, up to cyclic permutations. A parameter which is proportional to the excluded volume for the particles is the average size of the molecules \tilde{s} given by Eq. (60). In the case of the simple exclusion problem, all the molecules have the same unity size $s_1 = s_2 = \dots = s_n = 1$, which gives $\tilde{s} = 1$. On the other hand, if all the particles have no size, we have $\tilde{s} = 0$ and there is no excluded volume. By choosing a suitable distribution $\{s\}$ of the molecules' sizes, we may change \tilde{s} almost continuously in the bulk limit $N \rightarrow \infty$. Exploiting the connection between the dynamical critical exponent z and the mass gap of the related quantum chain, we obtained $z = \frac{3}{2}$ for all the models, independently of the parameter \tilde{s} measuring the excluded volume. This implies that, at least in one dimension, the excluded volume effect is irrelevant for dynamical systems in the KPZ universality class.

We also show (see the Appendix) that the wave functions of the models with an arbitrary distribution of the molecules' sizes can be related to those of a simple asymmetric exclusion problem, in a distinct lattice size. This implies that conditional probabilities and correlation functions of these models are exactly related.

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APPENDIX A: EXACT SPECTRAL RELATIONS BETWEEN THE MODELS WITH DISTINCT MOLECULE DISTRIBUTION

In this appendix we are going to show how the eigenvalues and eigenvectors of the general Hamiltonian (7) with a

different distribution of molecules are related to those of the simple asymmetric exclusion Hamiltonian (1). Let us consider initially the case of free boundary conditions. In this case we have to specify, for a given lattice size N , the occupation at the border of the lattice, i.e., the minimal x_I and maximum x_F coordinates in which we may put a molecule of size s . As an extension of Eq. (2) we define these coordinates, for the model with a distribution $\{s_1, s_2, \dots, s_n\}$ of molecules, as

$$\begin{aligned} x_I &= d_{s_1,0} + \delta_{s_1,0} = \text{Int}[(s_1 + 1)/2] + \delta_{s_1,0}, \\ x_F &= N - (d_{s_n,0} - 1) - \delta_{s_n,0}. \end{aligned} \quad (\text{A1})$$

If $|x_1, \dots, x_n\rangle$ are the vectors corresponding to the coordinates of the particles of sizes $\{s_1, \dots, s_n\}$, the application of the Hamiltonian (7) with free ends in a given vector can be written as

$$\begin{aligned} H_{\{s\}}^F |x_1, \dots, x_n\rangle &= - \sum_{i=1}^{n-1} \epsilon_+ \theta(x_{i+1} - x_i - d_{s_i, s_{i+1}}) \\ &\times [|x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n\rangle \\ &- |x_1, \dots, x_n\rangle] - \sum_{i=2}^n \epsilon_- \theta(x_i - x_{i-1} \\ &- d_{s_{i-1}, s_i}) [|x_1, \dots, x_{i-1}, x_i \\ &- 1, x_{i+1}, \dots, x_n\rangle - |x_1, \dots, x_n\rangle] \\ &- \epsilon_+ \theta(x_F - x_n) [|x_1, \dots, x_{n-1}, x_n + 1\rangle \\ &- |x_1, \dots, x_n\rangle] - \epsilon_- \theta(x_1 - x_I) \\ &\times [|x_1 - 1, x_2, \dots, x_n\rangle - |x_1, \dots, x_n\rangle], \end{aligned} \quad (\text{A2})$$

where $\theta(x) = 0$ for $x \leq 0$ and $\theta(x) = 1$ for $x > 0$ is the standard step function.

If we now make the change of coordinates

$$x'_i = x_i - x_I - \sum_{j=1}^{i-1} d_{s_j, s_{j+1}} + i, \quad (\text{A3})$$

we can rewrite Eq. (A1) as

$$\begin{aligned} H_{\{s\}}^F |x'_1, \dots, x'_n\rangle &= - \sum_{i=1}^{n-1} \epsilon_+ \theta(x'_{i+1} - x'_i - 1) \\ &\times [|x'_1, \dots, x'_{i-1}, x'_i + 1, \dots, x'_n\rangle \\ &- |x'_1, \dots, x'_n\rangle] - \sum_{i=2}^n \epsilon_- \theta(x'_i - x'_{i-1} - 1) \\ &\times [|x'_1, \dots, x'_{i-1}, x'_i - 1, x'_{i+1}, \dots, x'_n\rangle \\ &- |x'_1, \dots, x'_n\rangle] - \epsilon_+ \theta(x'_F - x'_n) \\ &\times [|x'_1, \dots, x'_{n-1}, x'_n + 1\rangle - |x'_1, \dots, x'_n\rangle] \\ &- \epsilon_- \theta(x'_1 - x'_I) [|x'_1 - 1, x'_2, \dots, x'_n\rangle \\ &- |x'_1, \dots, x'_n\rangle], \end{aligned} \quad (\text{A4})$$

where now $x'_I = 1$ and $x'_F = x_F - x_I - \sum_{j=1}^{n-1} d_{s_j, s_{j+1}} + n \equiv N'$. But this is exactly the application of the simple asymmetric exclusion Hamiltonian $H_{\{s_1 = \dots = s_n = 1\}}^F$ in a lattice size N' . Consequently, for free ends there exists a one-to-one correspondence between the eigenvalues and eigenvectors of our general Hamiltonian (7) with an arbitrary distribution of particle sizes with that of the standard simple diffusion problem, in a lattice size which depends on the volume excluded due to the molecules' sizes, i.e.,

$$H_{\{s\}}^F(n, N) = H_{\{s_1 = s_2 = \dots = s_n = 1\}}^F(n, N'), \quad (\text{A5})$$

$$N' = N - \sum_{j=1}^{n-1} d_{s_j, s_{j+1}} - d_{s_1, 0} - d_{s_n, 0} + n + 1 - \delta_{s_1, 0} - \delta_{s_n, 0}. \quad (\text{A6})$$

The Bethe-ansatz solution of the XXZ chain with surface fields given in [20], after the canonical transformation (11), can be easily exported for our general model (7) with free ends. As observed in [21], the simple asymmetric diffusion Hamiltonian (1), with free boundaries, has a quantum $U_q SU(2)$ symmetry with $q = \sqrt{\epsilon_+ / \epsilon_-}$. This symmetry implies an exact form for the ground-state wave function. Using this wave function in the relation (A3) we obtain the ground-state wave function for the general Hamiltonian (7) with free ends,

$$\Psi_0^{\{s\}} = \sum_{\{x\}} \prod_{i=1}^n \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}^{x_i + i - \sum_{j=1}^{i-1} d_{s_j, s_{j+1}}} |x_1, x_2, \dots, x_n\rangle. \quad (\text{A7})$$

Let us now consider the case of the Hamiltonian (7) with twisted boundary conditions specified by the angle ϕ , which in general is a complex number,

$$\begin{aligned} E_{N+1}^{\beta, 0} &= e^{i\phi} E_1^{\beta, 0}, & E_{N+1}^{0, \beta} &= e^{-i\phi} E_1^{0, \beta}, \\ \beta \neq 0 E_{n+1}^{0, 0} &= E_1^{0, 0}, & E_{N+1}^{\beta, \beta} &= E_1^{\beta, \beta}. \end{aligned} \quad (\text{A8})$$

The periodic case treated in Sec. III corresponds to the case where $\phi = 0$. The Bethe-ansatz equations for these boundary conditions can be obtained by changing Eqs. (52) and (53), and are given by

$$e^{ik_j N'} e^{i\tilde{\phi}_m} = (-1)^{n-1} \prod_{l=1}^n \frac{\epsilon_+ + \epsilon_- e^{i(k_j + k_l)} - e^{ik_j}}{\epsilon_+ + \epsilon_- e^{i(k_j + k_l)} - e^{ik_l}}, \quad (\text{A9})$$

where

$$\begin{aligned} N' &= N - n(\tilde{s} - 1), & \tilde{\phi}_m &= \phi + P(\tilde{s} - 1) + \frac{2\pi}{r} m, \\ m &= 0, 1, \dots, r-1, \end{aligned} \quad (\text{A10})$$

$$P = \sum_j k_j \bmod(2\pi) = \frac{2\pi}{N} l, \quad l = 0, 1, \dots, N-1 \quad (\text{A11})$$

is the momentum and r, \tilde{s} are defined in Eqs. (57)–(60). These equations give us the following equivalence between

the eigenspectra of the general Hamiltonian (7) with boundary condition ϕ , in the eigensector with n particles and a given momenta P :

$$H_{\{s_1, \dots, s_n\}}^\phi(N, P, n) = \sum_{m=0}^{r-1} H_{\{s_1=s_2=\dots=s_n=1\}}^{\tilde{\phi}_m}(N', P, n), \quad (\text{A12})$$

where in the right-hand side we have the eigenspectra of the asymmetric simple exclusion Hamiltonian (1) with twisted boundary condition $\tilde{\phi}_m$. In the right-hand side of the above equation we must also add several eigenspectra, depending on the value of r (see Sec. III), and this is due to the distinguishability of the particles in the Hamiltonian on the left-hand side of the equation.

Our Bethe solutions presented in Sec. III also give us a connection between the wave functions of the models. The eigenstates related by Eq. (A12), apart from an overall normalization, satisfy

$$\begin{aligned} \Psi_{\{s_1, \dots, s_n\}}^{\{k_1, \dots, k_n\}}(x_1, x_2, \dots, x_n) \\ = \Psi_{\{s_1=s_2=\dots=s_n=1\}}^{\{k_1, \dots, k_n\}}(x'_1, x'_2, \dots, x'_n), \end{aligned} \quad (\text{A13})$$

where

$$x'_1 = x_1, \quad x'_i = x_i + i - 1 - \sum_{j=1}^{i-1} d_{s_j, s_{j+1}}, \quad i=2, \dots, n. \quad (\text{A14})$$

The results (A5), (A13), and (A14) imply that any calculation involving only wave functions can be straightforwardly translated for arbitrary distribution $\{s\}$ of the molecules' sizes. An example of this is the n -particle Green's function

$P^{\{s\}}(x_1, \dots, x_n; t | y_1, \dots, y_n; 0)$, which gives the probability of finding particles of size s_i , initially ($t=0$) at y_i and at time t at x_i ($i=1, \dots, n$). These Green's functions for the different models satisfy

$$\begin{aligned} P^{\{s_1, \dots, s_n\}}(x_1, \dots, x_n; t, y_1, \dots, y_n; 0) \\ = P^{\{s_1, \dots, s_n=1\}}(x'_1, \dots, x'_n; t, y'_1, \dots, y'_n; 0), \end{aligned} \quad (\text{A15})$$

where x'_i and y'_i are related to x_i and y_i as in Eq. (A3). The above result generalizes that obtained by Sasamoto and Wadati [12] for the case in which we have a fully asymmetric model ($\epsilon_- = 0$) and molecules of identical sizes $s_1 = s_2 = \dots = s_n = s$.

Calculation involving eigenvalues, like the calculation of the exponent z we did in Sec. IV for the fully asymmetric model ($\epsilon_- = 0$), should be translated with care among the different models. The Bethe-ansatz equations (A9) tell us that the eigenvalues of our general model (7) in a periodic lattice of size N ($\phi=0$) are the same as those of the simple exclusion Hamiltonian (1) in a lattice of size $N' = N - n(\tilde{s} - 1)$ and twisted boundary condition $\tilde{\phi}_m = P(\tilde{s} - 1) + (2\pi/r)m$, $m=0, 1, \dots, r-1$. However, the eigenvalues of the simple asymmetric exclusion Hamiltonian (1) depend on the boundary condition. Our results of Sec. III, although valid only for $\epsilon_- = 0$, indicate that the effect of the boundary angle in the finite-size corrections is of higher order than the leading corrections for the first excited state, since in this case $\tilde{\phi}_m = \tilde{\phi}_0 = 2\pi(\tilde{s} - 1)/N$. This implies that for an arbitrary distribution of molecules and arbitrary values of ϵ_+ , ϵ_- and densities we can use the results obtained in [10], for the leading order of the real part of the mass gap, which gives a universal dynamical critical exponent $z = \frac{3}{2}$ of KPZ-type.

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